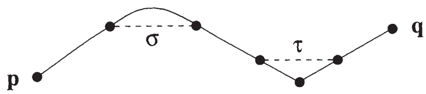

FIG. 8.6

neighborhood of $\alpha_i(t_0)$. Note that $\alpha_i|[t_0, t_0 + \varepsilon]$ cannot be reparametrized to be geodesic. Thus by the uniqueness feature of Theorem 1.8, the radial geodesic segment σ from $\alpha_i(t_0)$ to $\alpha_i(t_0 + \varepsilon)$ is strictly shorter (Fig. 8.6). Replacing $\alpha_i|[t_0, t_0 + \varepsilon]$ by σ changes α to a strictly shorter curve from **p** to **q**, contradicting the assumption that α is shortest. Thus α is a possibly broken geodesic.

Now we assume that α actually has a corner, say at $\alpha_{i-1}(b_i) = \alpha_i(b_i)$, and again deduce a contradiction. By the remark preceding this lemma, there is an $\varepsilon > 0$ such that $\alpha_{i-1}(b_i)$ is contained in in a normal neighborhood $\mathcal{N}$ of $\alpha_{i-1}(b_i - \varepsilon)$. By continuity, some initial subsegment $\alpha_i|_{[b_i, t_1]}$ of α_i is still in $\mathcal{N}$.

Thus the combined curve from $\alpha_{i-1}(b_i - \varepsilon)$ to $\alpha_i(t_1)$ has a corner and lies in $\mathcal{N}$. So it is strictly longer than the radial geodesic τ of $\mathcal{N}$ joining these points (Fig. 8.6). Then as before, replacing the combined curve by τ shortens α, giving the required contradiction. ◆

Exercises

1. Prove:
(a) A normal ε-neighborhood of **p** in M consists of all points **q** in M such that $\rho(\mathbf{p}, \mathbf{q}) < \varepsilon$.
(b) If $\mathbf{p} \neq \mathbf{q}$ in M, then $\rho(\mathbf{p}, \mathbf{q}) > 0$. (This completes the proof that intrinsic distance ρ is a metric on M; see Ex. 3 of Sec. 6.4.)

2. (*Normal coordinates.*) Let $\mathcal{N} = \exp_p(\mathcal{U})$ be a normal neighborhood of a point **p** in M, and let $\mathbf{e}_1$, $\mathbf{e}_2$ be a frame at **p**. Prove:
(a) The mapping

$$\mathbf{n}(x, y) = \exp_p(x\mathbf{e}_1 + y\mathbf{e}_2)$$

is a coordinate patch on $\mathcal{N}$.

(b) At **p** (but generally not elsewhere) $E = 1$, $F = 0$, $G = 1$. Thus normal coordinates are Euclidean at **p**, hence at least approximately Euclidean near **p**.
(c) Coordinate straight lines through **p** are geodesics of M.

(d) With suitable choices, $\mathbf{n}$ for $\mathbf{R}^2$ is the identity map $\mathbf{n}(x, y) = (x, y)$. So for arbitrary M, normal coordinates generalize the natural (rectangular) coordinates of $\mathbf{R}^2$.

3. At the point $\mathbf{p} = (r, 0, 0)$ of the cylinder M: $x^2 + y^2 = r^2$, let $\mathbf{e}_1 = (0, 1, 0)$ and $\mathbf{e}_2 = (0, 0, 1)$. Find an explicit formula for the normal parametrization in Exercise 2. What is the largest normal neighborhood of the point $\mathbf{p}$?

4. (*Continuation.*) Prove:

(a) A geodesic starting at an arbitrary point $\mathbf{p} = (a, b, c)$ in the cylinder M does not minimize arc length after it passes through the antipodal line $t \to (-a, -b, t)$. (Only vertical geodesics through $\mathbf{p}$ fail to meet this line.)

(b) If $\mathbf{q}$ is not on the antipodal line of $\mathbf{p}$, there is a unique shortest geodesic from $\mathbf{p}$ to $\mathbf{q}$.

(c) Derive a formula for intrinsic distance on the cylinder.

5. Let M be an augmented surface of revolution (Ex. 12 of Sec. 4.1). Prove, without computation:

(a) If M has only one intercept $\mathbf{p}$ on the axis of revolution, then every geodesic segment γ starting at $\mathbf{p}$ uniquely minimizes arc length.

(b) If M has a second intercept $\mathbf{q}$, then the conclusion in (a) holds if and only if γ does not reach $\mathbf{q}$.

6. In M let α be a curve segment in M from $\mathbf{p}$ to $\mathbf{q}$, and β a curve segment from $\mathbf{q}$ to $\mathbf{r}$. Joining α and β does not usually produce a differentiable curve from $\mathbf{p}$ to $\mathbf{r}$ since $\alpha + \beta$ will usually have a corner at $\mathbf{q}$.

In this case, prove that there is a piecewise smooth curve γ from $\mathbf{p}$ to $\mathbf{r}$ that is arbitrarily close to $\alpha + \beta$ but strictly shorter: $L(\gamma) < L(\alpha) + L(\beta)$. (*Hint:* See proof of Cor. 1.10.)

Techniques from advanced calculus show that the corner can actually be smoothed away, leaving γ differentiable throughout.

7. (*Intrinsic distance is continuous.*)

(a) For $\mathbf{p}_0$ in M, show that the real-valued function $\mathbf{p} \to \rho(\mathbf{p}_0, \mathbf{p})$ is continuous; in fact, if $\rho(\mathbf{p}, \mathbf{q}) < \varepsilon$, then $|\rho(\mathbf{p}_0, \mathbf{p}) - \rho(\mathbf{p}_0, \mathbf{q})| < \varepsilon$.

(b) State precisely and prove that the function $(\mathbf{p}, \mathbf{q}) \to \rho(\mathbf{p}, \mathbf{q})$ is continuous.

8. The radial geodesics from the point (0, 1) in the Poincaré half-plane P are given by $(x - a)^2 + y^2 = a^2 + 1$ for all a (see Ex. 1 of Sec. 7.5).

(a) Show that the curves given by $x^2 + (y - b)^2 = b^2 - 1$ for all $b > 1$ are everywhere orthogonal to these geodesics.

(b) Deduce that the curves in (a) are the polar circles about the pole (0, 1).

(c) (*Computer graphics.*) Plot a few curves from each family on the same figure.